A Relation between Group Order of Elliptic Curve and Extension Degree of Definition Field

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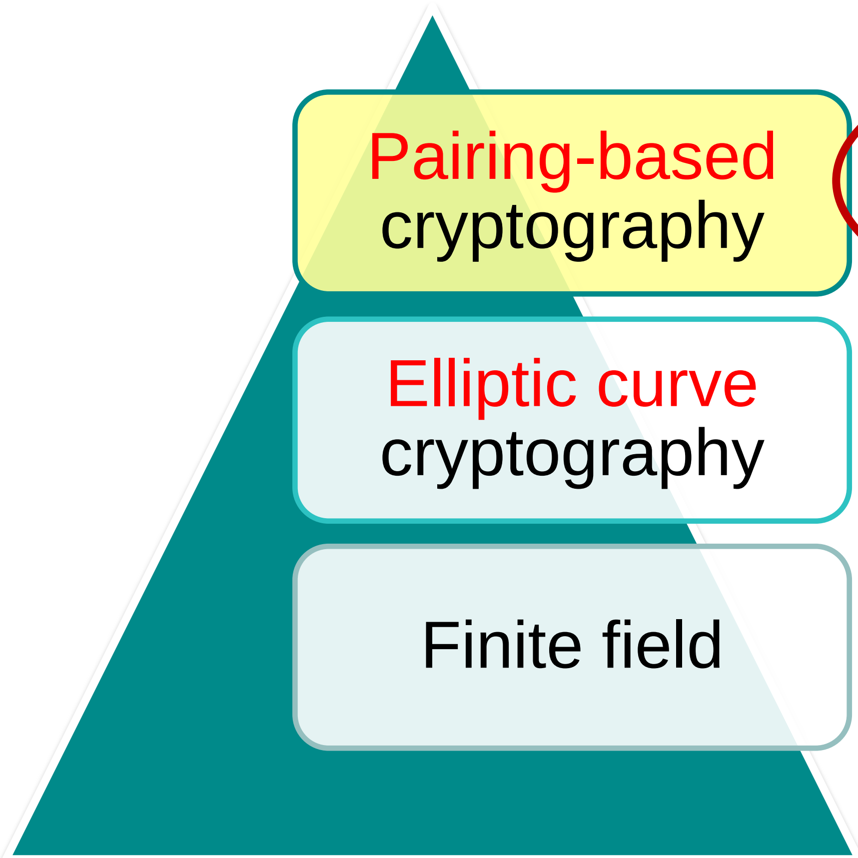
Research Background

Recent innovative cryptographic applications are based on …



- **ID-based cryptography**
- **Group signature authentication**
- **Time release cryptography**

- **EC discrete logarithm problem**
  - **Elliptic curve addition**

- **Prime field and Extension field**
- **Arithmetic operations**
  - **Addition/Subtraction**
  - **Multiplication/Division**

Research Background

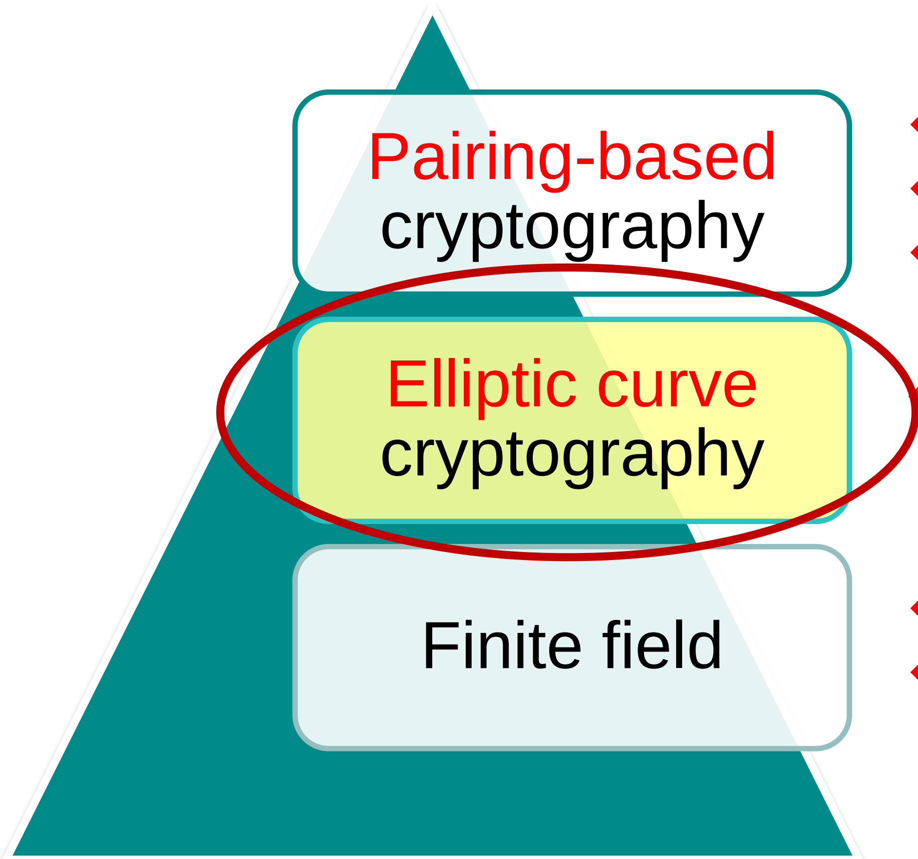
- ID-based cryptography
  - We can use ID-based information as public key.
    - User name
    - E-main address
    - Phone number etc.

- Group signature authentication
  - Anonymous authentication
  - Attribute-based authentication

- Time release cryptography
  - It keeps the data encrypted until the day for release comes.

Research Background

Pairing-based cryptography is based on elliptic curve cryptography.



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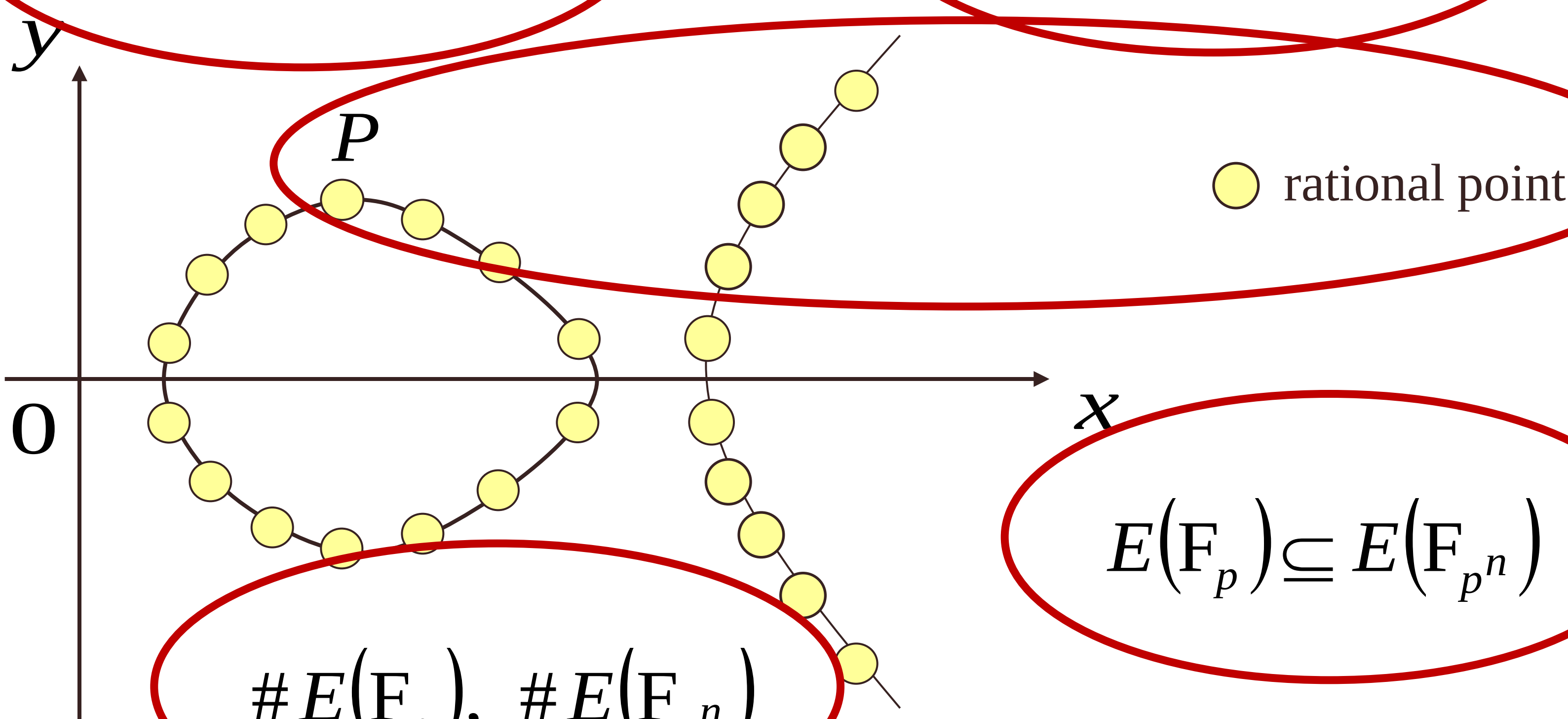
  - **Addition**
  - **Multiplication**

*Finite and discrete mathematics

Mathematical notations

$$E : y^2 = x^3 + ax + b, \quad a, b, x, y \in \mathrm{F}_{p^n} \qquad \mathrm{F}_p \subseteq \mathrm{F}_{p^n}$$



$$E(\mathrm{F}_p) \subseteq E(\mathrm{F}_{p^n})$$

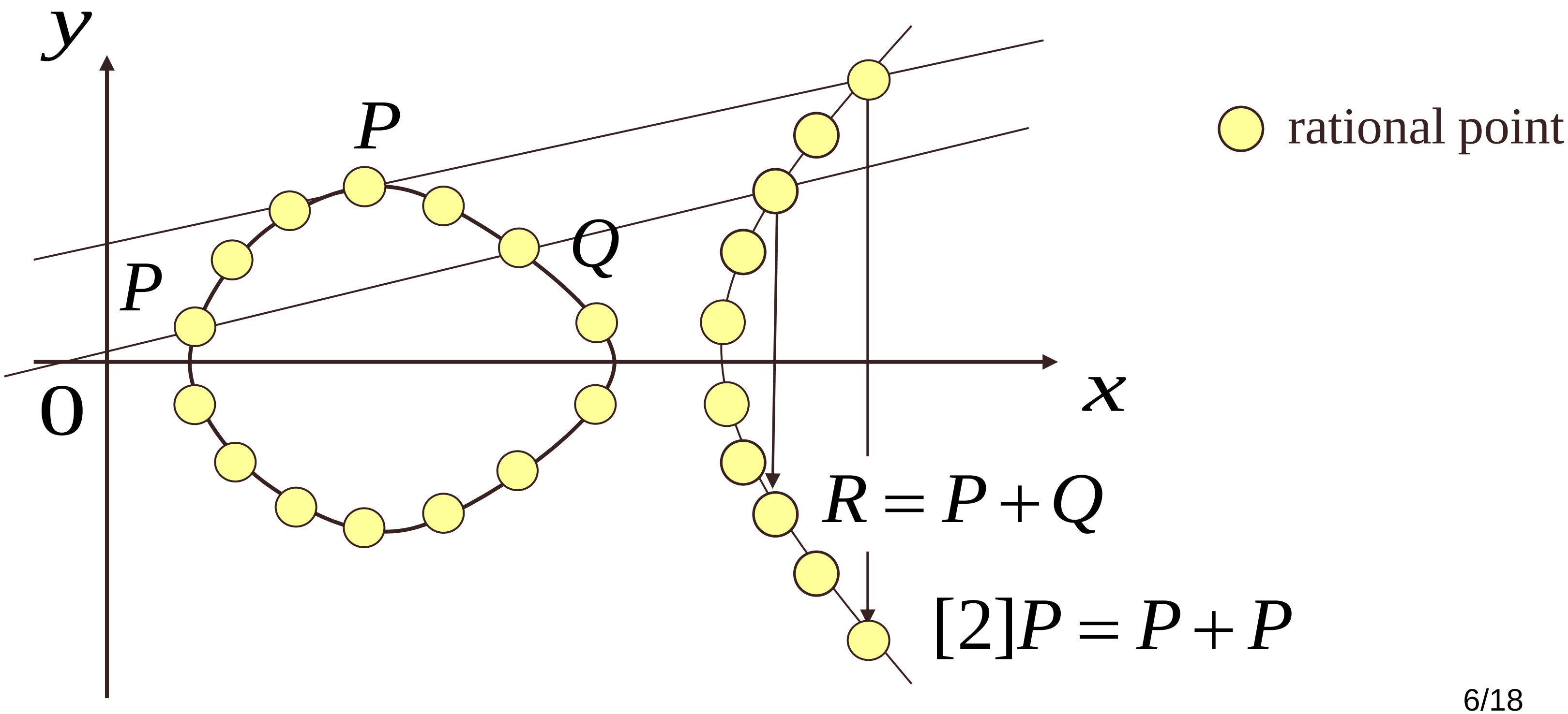
$$\# E(\mathrm{F}_p), \ \# E(\mathrm{F}_{p^n})$$

Elliptic curve cryptography (1/2)

Elliptic curve cryptography

Arithmetic operations

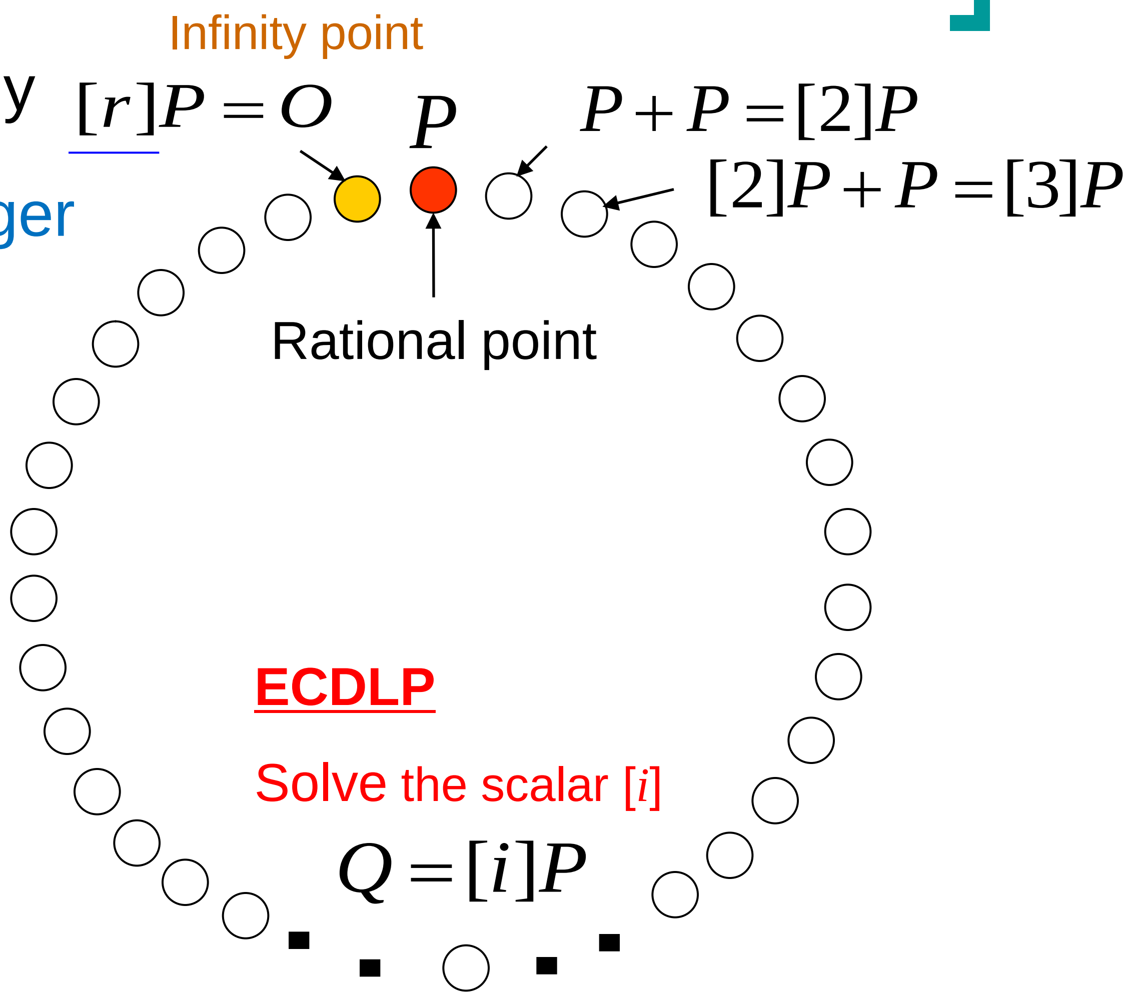
$$E: y^2 = x^3 + ax + b, \quad a, b, x, y \in \mathrm{F}_{p^n}$$



Elliptic curve cryptography (2/2)

Elliptic curve cryptography

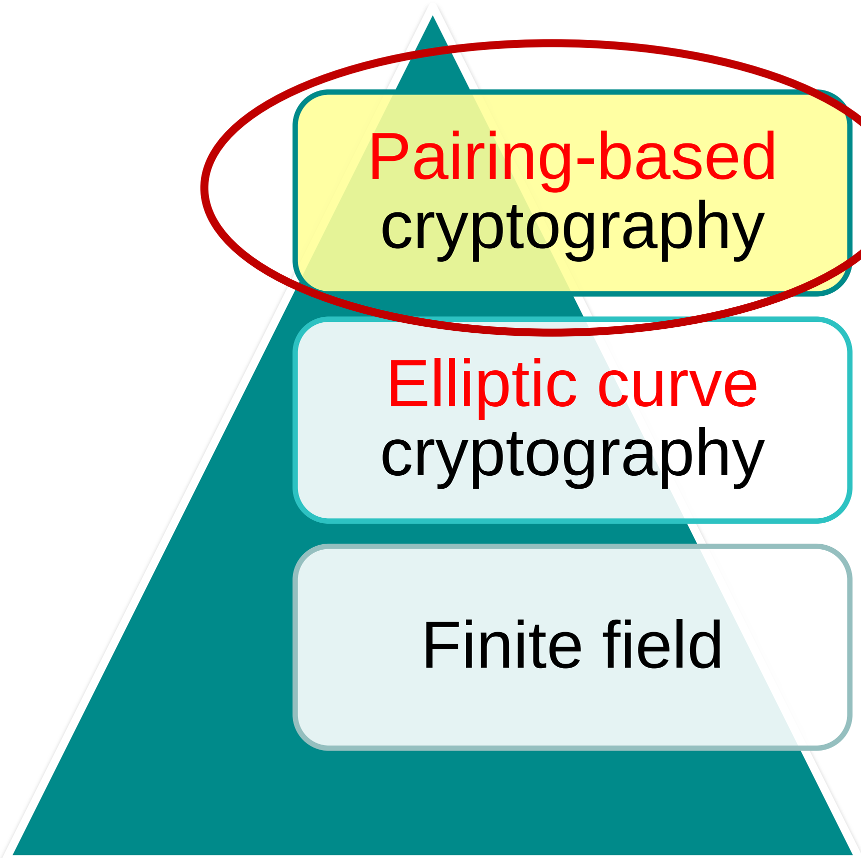
the order r is larger than 160 bits



Cyclic group

Research Background

Pairing-based cryptography uses a special class of elliptic curve.



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Pairing-based cryptography (1/3)

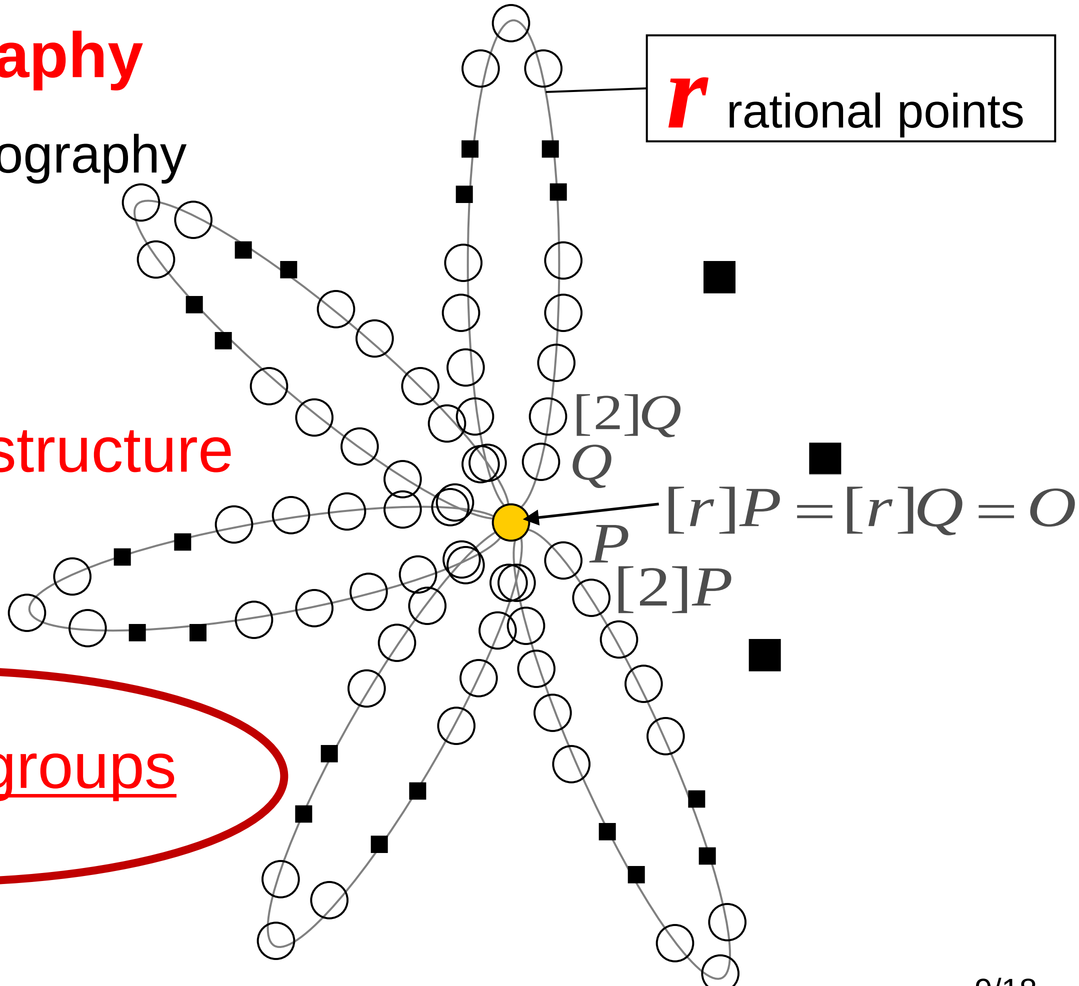
Pairing-based cryptography

over elliptic curve cryptography

Requirement:
torsion group structure

Pairing uses two cyclic groups

among $r+1$



Pairing-based cryptography (2/3)

- Pairing-friendly curves
  - It is defined over extension field F_{p^n} ***n*-th vector space**
  - The defining equation is

$$E: y^2 = x^3 + ax + b, \quad a, b, x, y \in F_{p^n}$$

- Some conditions should be satisfied
  - Torsion group structure
  - The number of rational points $\#E(F_{p^n})$

needs to be divisible by r^2.

Pairing-based cryptography (3/3)

- How to prepare pairing-friendly curves
  - It is difficult except for some special curves
    - Barreto-Naehrig (BN) curve : $n = 12$
    - Brezing-Weng (BW) curve : $n = 8$

- Setting parameters :
  - ***p, a, b***
  - #rational points ***r***
  - dimension ***n***

$$E : y^2 = x^3 + ax + b$$

$$a, b, x, y \in \mathrm{F}_{p^n}$$

$$\#E\left(\mathrm{F}_{p^n}\right) \quad r$$

Target of this research

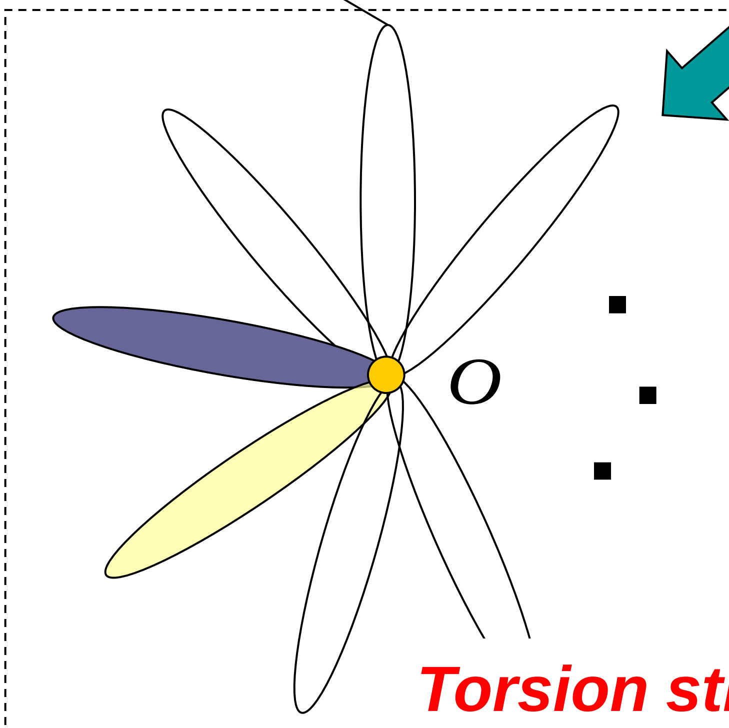
There are r points in each cyclic group

n : the extension degree (dimension)

$$n \mid (r-1)$$

There are some previous works.

BN and BW curves



Torsion structure

$$n \nmid (r-1)$$

There are few reports

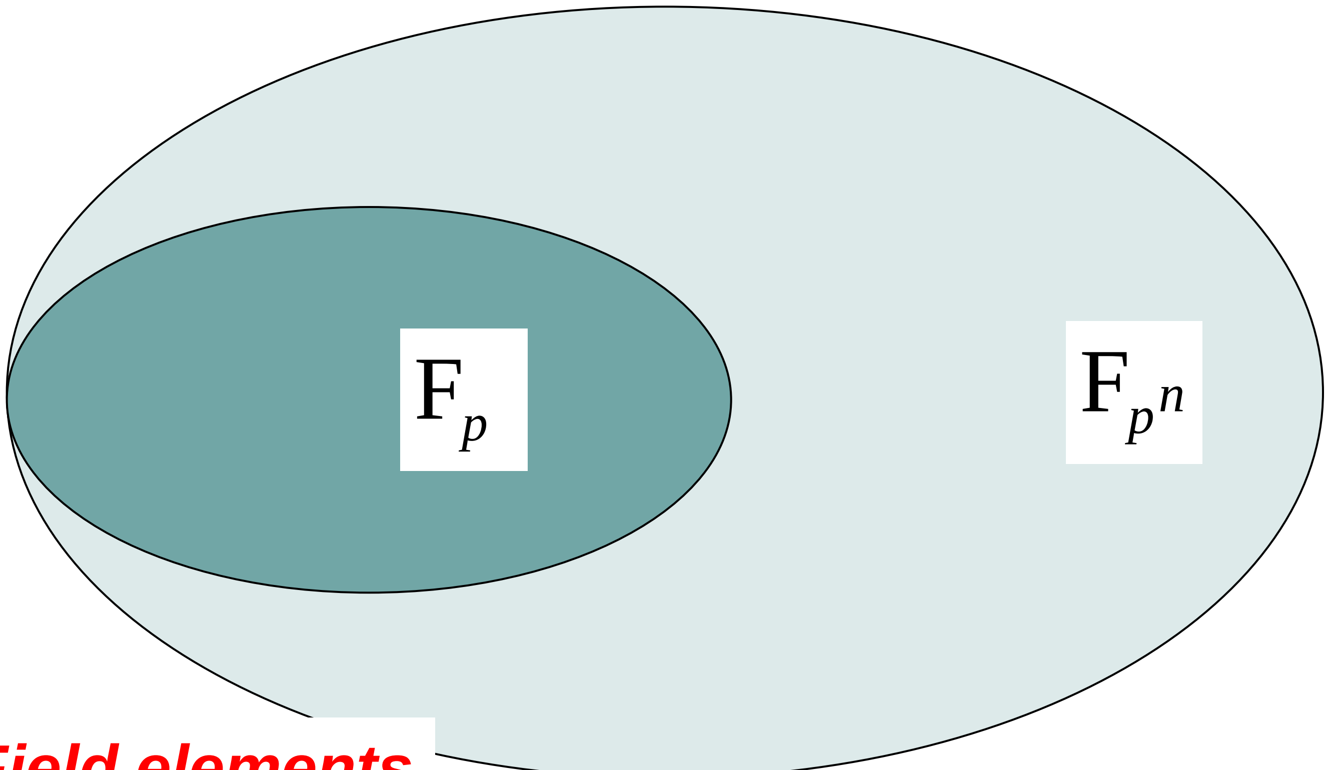
$$n \mid (r+1)$$

$$n = r$$

The target of this work

Algebraic closure

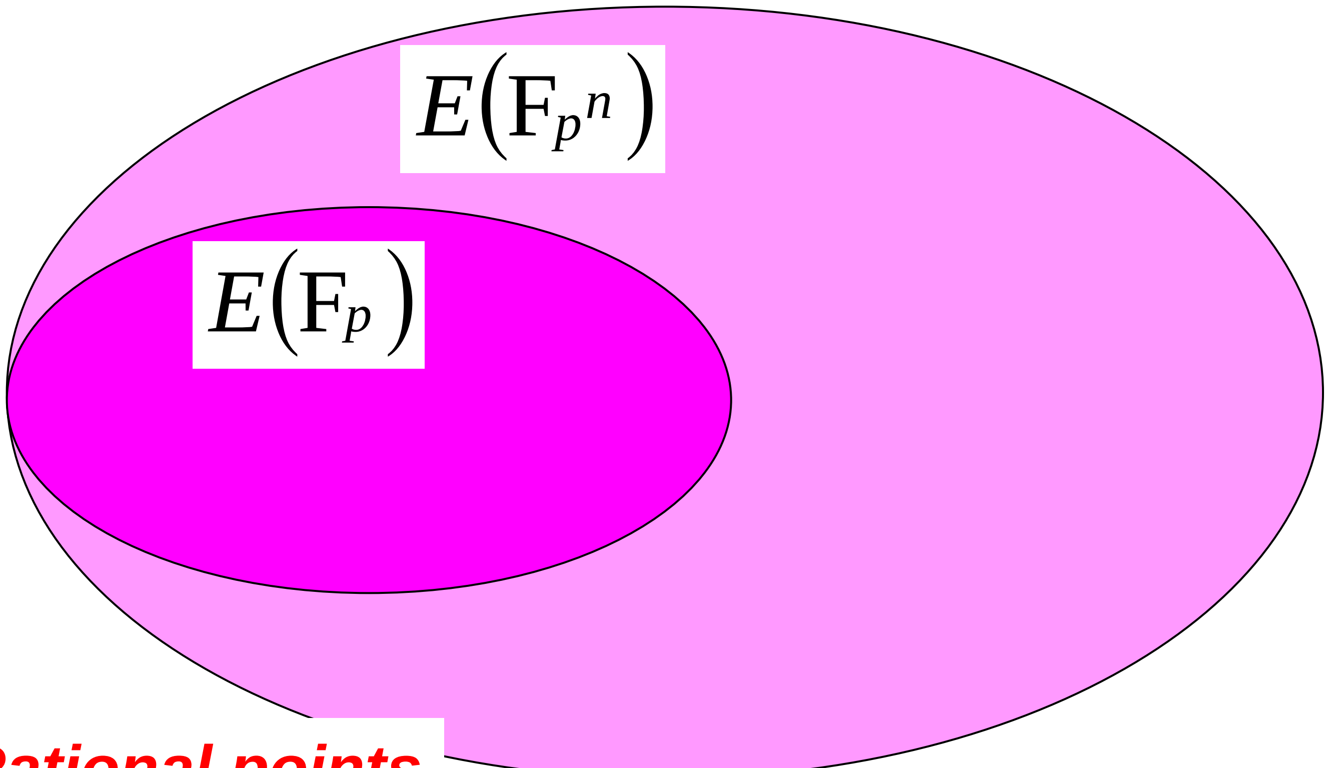
- Prime field F_p and n-th Extension field F_{p^n}



: Field elements

In the same Elliptic curve closure

- Over Prime field $E(\mathrm{F}_p)$ and ex. field $E(\mathrm{F}_{p^n})$

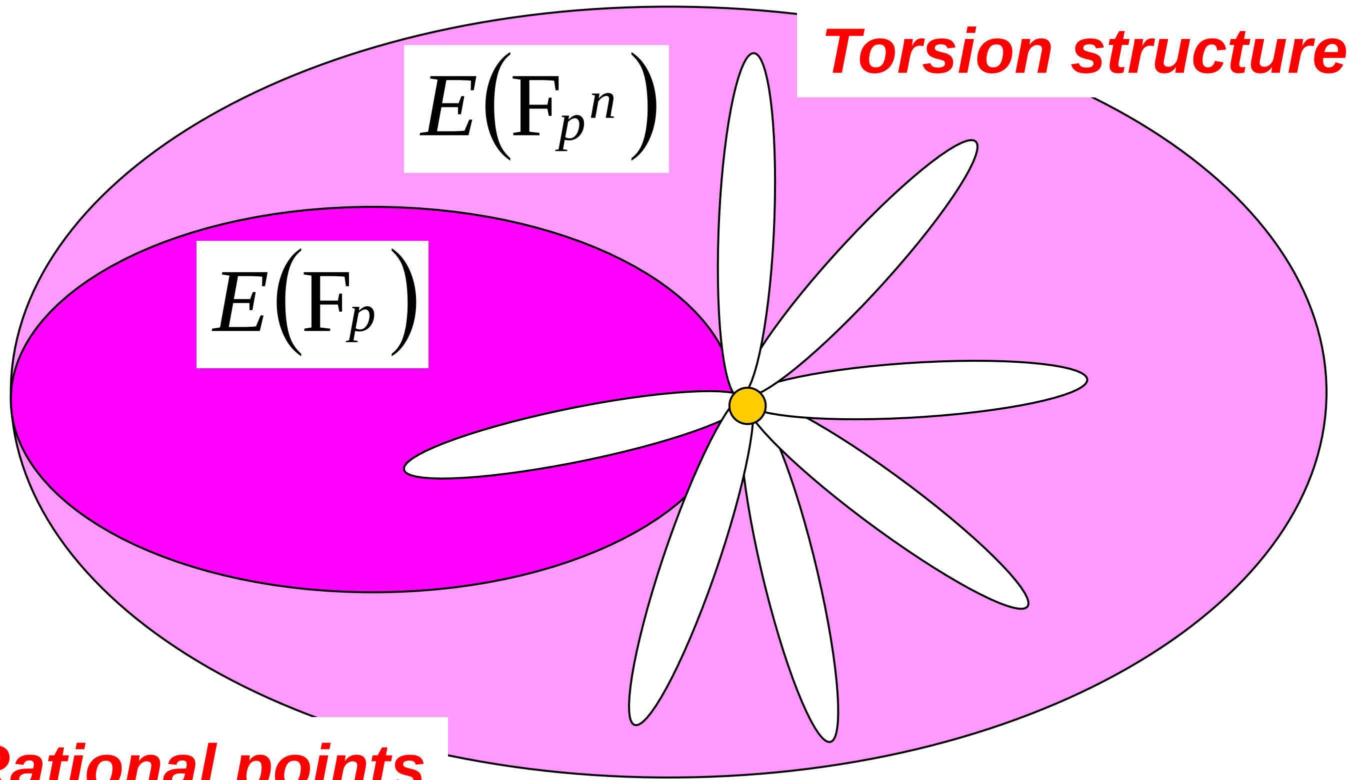


: Rational points

Our contribution (theoretic proof was given)

- <u>If</u> $r \mid \# E(\mathrm{F}_p)$ <u>and</u> $n = r$

Torsion structure appears



: Rational points

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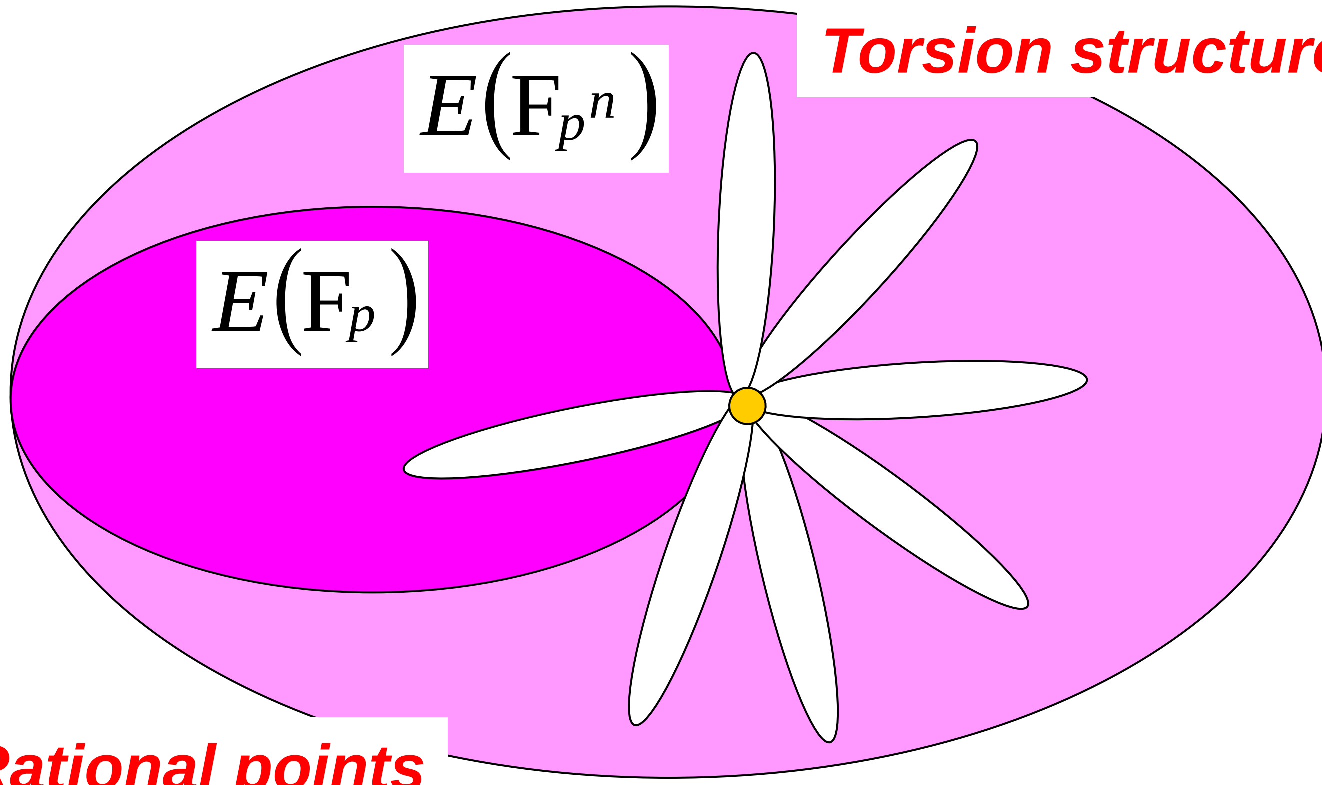

Example

Example 1:

$$p = 11, \quad r = 5,$$
$$E : \; y^2 = x^3 + 6x + 3,$$
$$\#E(\mathbb{F}_p) = 15, \quad \#E(\mathbb{F}_{p^5}) = 161625.$$

Conclusion

- This work has focused on $n = r$
  - Torsion structure appears

- Further consideration
  - Consider pairing-based cryptographic applications.

Thank you for your attentions.